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Liouville Perturbation Theory*

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Abstract

A comparison is made between proposals for the exact three point function in Liouville quantum field theory and the nonperturbative weak coupling expansion developed long ago by Braaten, Curtright, Ghandour, and Thorn. Exact agreement to the order calculated (*i.e.* up to and including corrections of order $O(g^{10})$) is found.

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1 Introduction

Much progress has been made in our understanding of Liouville quantum field theory since Polyakov discovered its relevance to subcritical string theory [1]. In particular, bootstrap ideas have led to proposals for the exact three point correlators [2, 3]

$$\langle e^{2\alpha_1\phi(x_1)}e^{2\alpha_2\phi(x_2)}e^{2\alpha_3\phi(x_3)}\rangle = |x_{12}|^{2\gamma_3}|x_{23}|^{2\gamma_1}|x_{31}|^{2\gamma_2}C(\alpha_1,\alpha_2,\alpha_3) \tag{1}$$

of the exponentials of the Liouville field, whose dynamics is given by the Lagrangian

$$\mathcal{L} = -\frac{1}{4\pi}(\partial\phi)^2 - \mu e^{2b\phi}. \tag{2}$$

Here the Liouville field theory is imagined to be defined on a two dimensional sphere.

Many years ago, a systematic weak coupling expansion for the Liouville theory, defined on a flat cylinder and quantized using standard canonical methods, was developed by Braaten, Curtright, Ghandour, and Thorn [5]. In this treatment the zero mode of the Liouville field was treated exactly by its Liouville quantum mechanics, but the nonzero modes were treated perturbatively. In this way a weak coupling ($b \to 0$) expansion to any finite order could be developed for matrix elements of products of an arbitrary number of operators $e^{2b\sigma_i\phi(x_i)}$ between energy eigenstates. In particular, the matrix element of $e^{2b\sigma\phi}$ between energy eigenstates with energies of order $O(b^2)$ was expanded up to and including $O(b^{10})$ terms. Since this matrix element is directly related to the three point correlator for which an exact answer has been proposed, we thought it would be useful to check whether the exact proposal agreed with the much older weak coupling calculation. In this brief note we show that there is exact detailed agreement to the order calculated. This is important, first because it gives another piece of evidence in favor of the proposed exact formula, and secondly because it confirms the reliability of standard canonical methods in the quantization of quantum field theory.

The matrix element of a product of $n-2$ exponentials on the cylinder is related to the correlator of n exponentials on a sphere by associating the states with two points on the sphere represented on the complex plane by, say $x_1 = 0$ and $x_n = \infty$. Putting $\alpha_1 = Q/2 + iP$ and $\alpha_n = Q/2 + iP'$, with P, P' real, the energies of the two states participating in the matrix element are given by $E = 2P^2$ and $E' = 2P'^2$. Then the matrix elements calculated in Ref. [5] were those of the operator $e^{2b\sigma\phi}$ between states with $P = bk$ and $P' = bk'$, and they were expanded in an asymptotic series in the limit $b \to 0$. In the following section we first develop the special function $\Upsilon(b\sigma)$, which figures in the proposed formula for the exact three point function, in an asymptotic series for $b \to 0$ with σ fixed. Using this result in the exact formula, we then expand the proposed three point function to $O(b^{10})$ and find complete agreement with the results of [5].

2 The Comparison

2.1 The Upsilon Function

The construction of the Liouville three point function [2, 3] (for a recent review of Liouville Theory, see [6]) employs the the special function $\Upsilon(x)$ which satisfies the functional relations

$$\Upsilon(x+b) = \frac{\Gamma(bx)}{\Gamma(1-bx)}b^{1-2bx}\Upsilon(x) \tag{3}$$

$$\Upsilon(x+1/b) = \frac{\Gamma(x/b)}{\Gamma(1-x/b)}b^{-1+2x/b}\Upsilon(x). \tag{4}$$

To analyze the limit $b \to 0$ define $g(x) = b^{x^2-bx}\Upsilon(x)$. Then it is appropriate to focus on the first of these relations which becomes in terms of g

$$g(x+b) = \frac{b\Gamma(bx)}{\Gamma(1-bx)}g(x). \tag{5}$$

First examine the $b \to 0$ limit with $\sigma = x/b$ fixed. Define $f(\sigma) = b^\sigma \Gamma(\sigma) g(b\sigma)$ so we have

$$\begin{aligned} f(\sigma+1) &= \frac{\Gamma(1+b^2\sigma)}{\Gamma(1-b^2\sigma)}f(\sigma) &(6)\\ &= f(\sigma)\exp\left\{-2\gamma b^2\sigma - 2\sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}(b^2\sigma)^{2n+1}\right\}. &(7)\end{aligned}$$

We can develop a solution of this equation in perturbation theory by making the *ansatz*

$$f(\sigma) = f_0 \exp\left\{b^2\gamma\phi_1(\sigma) + \sum_{n=1}^{\infty} b^{2(2n+1)}\frac{\zeta(2n+1)}{2n+1}\phi_{2n+1}(\sigma)\right\} \tag{8}$$

with $\phi_n(\sigma)$ a polynomial of order $n+1$, vanishing at $\sigma = 0$ and satisfying the relation

$$\phi_n(\sigma+1) \quad = \quad \phi_n(\sigma) - 2\sigma^n \tag{9}$$

A generating function for the ϕ's is

$$\sum_{n=0}^{\infty}\frac{\eta^n}{n!}\phi_n(\sigma) \equiv -2\frac{e^{\eta\sigma}-1}{e^\eta - 1}. \tag{10}$$

Since this generating function has a finite radius of convergence, determined by the singularities closest to the origin ($\eta = \pm 2\pi i$), it follows that the $\phi_n(\sigma)$ grow like $n!$ so the perturbation series diverges for finite b and is only an asymptotic series. We list the first 3 ϕ_{2n+1}:

$$\begin{aligned} \phi_1(\sigma) &= \sigma(1-\sigma) &(11)\\ \phi_3(\sigma) &= -\frac{1}{2}\sigma^2(1-\sigma)^2 &(12)\\ \phi_5(\sigma) &= -\frac{1}{6}\sigma^2(1-\sigma)^2(2\sigma^2-2\sigma-1) &(13)\end{aligned}$$

Summarizing, we have obtained the asymptotic expansion, for $b \to 0$ and σ fixed:

$$\Upsilon(b\sigma) = b\Upsilon_0\frac{b^{-b^2\sigma^2+(b^2-1)\sigma}}{\Gamma(\sigma)}\exp\left\{b^2\gamma\phi_1(\sigma) + \sum_{n=1}^{\infty} b^{2(2n+1)}\frac{\zeta(2n+1)}{2n+1}\phi_{2n+1}(\sigma)\right\} \tag{14}$$

where we have determined f_0 in terms of $\Upsilon_0 \equiv \Upsilon'(0)$ via $f_0 = b\Upsilon_0$

The conditions so far imposed to get this result would also be satisfied if (8) and hence (14) were multiplied by a periodic function $X(\sigma) = X(\sigma+1)$, with $X(0) = 1$[‡]. To show that in fact $X(\sigma) \equiv 1$, we refer to the theory of double gamma functions [4], which defines them by

$$\begin{aligned} \frac{d^3}{dz^3}\ln\Gamma_2(z|\omega_1,\omega_2) &\equiv -2\sum_{m,n=0}^{\infty}\frac{1}{(z+m\omega_1+n\omega_2)^3}\\ &= \frac{d^3}{dz^3}\ln\Gamma(z/\omega_1) - \int_0^\infty dt\frac{t^2e^{-zt}}{(e^{\omega_2 t}-1)(1-e^{-\omega_1 t})}. &(15)\end{aligned}$$

[‡]We thank J. Teschner for raising this issue

The function Υ is defined in terms of $\Gamma_2(z|b, 1/b)$. Clearly, this equation fixes $\ln \Gamma_2$ only up to an additive quadratic polynomial in z, which is fixed by the functional relations Γ_2 must satisfy. Nevertheless, setting $\omega_1 = b = 1/\omega_2$, the right side of (15) can be developed in an expansion for small $b^2 > 0$ which can be seen to be consistent with the detailed *ansatz* (8), up to an undetermined additive quadratic polynomial in the exponent. Since this *ansatz* led uniquely to (14), we conclude that $X(\sigma) = \exp(\alpha\sigma^2 + \beta\sigma)$. Periodicity in σ with period 1 implies then that $\alpha = 0$ and $\beta = 2\pi N i$ for some integer N. But since Υ is a real analytic function, we conclude that in fact $N = 0$ so (14) is indeed the correct asymptotic expansion for small b.

2.2 Formula for the Liouville Three Point Function

Ref. [2, 3] propose the following three point function for Liouville quantum field theory:

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) &= \left[\pi\mu b^{-2b^2} \frac{\Gamma(1+b^2)}{\Gamma(1-b^2)}\right]^{(Q-\sum \alpha_i)/b} \\ &\quad \times \frac{\Upsilon_0 \Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(-\alpha_1+\alpha_2+\alpha_3)\Upsilon(\alpha_1-\alpha_2+\alpha_3)} \end{aligned} \tag{16}$$

We would like to compare this formula with the perturbation expansion of Liouville on a cylinder given by Braaten, Curtright, Ghandour, and Thorn [5]. For that purpose two of the α's, say 1 and 3, must be of the form $iP + Q/2$ with P real. These would represent the states at $t = \pm\infty$. So write $\alpha_1 = ibk + Q/2$ and $\alpha_3 = ibk' + Q/2$. The third one is taken to be of $O(b)$: $\alpha_2 = b\sigma$. Then the arguments of all of the Υ's are small after exploiting the identity $\Upsilon(x) = \Upsilon(Q - x)$. We have

$$2\alpha_1 = Q + 2ibk \to -2ibk, \qquad 2\alpha_2 = 2b\sigma, \qquad 2\alpha_3 = Q + 2ibk' \to -2ibk' \tag{17}$$

$$\alpha_1 + \alpha_2 + \alpha_3 - Q = ib(k + k') + b\sigma \tag{18}$$

$$-\alpha_1 + \alpha_2 + \alpha_3 = ib(k' - k) + b\sigma \tag{19}$$

$$\alpha_1 + \alpha_2 - \alpha_3 = -ib(k' - k) + b\sigma \tag{20}$$

$$\alpha_1 - \alpha_2 + \alpha_3 = Q + ib(k' + k) - b\sigma \to -ib(k' + k) + b\sigma \tag{21}$$

Inserting these results into the formula gives

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) &= \left[\pi\mu b^{-2b^2} \frac{\Gamma(1+b^2)}{\Gamma(1-b^2)}\right]^{-\sigma-ik-ik'} \\ &\quad \times \frac{\Upsilon_0 \Upsilon(-2ibk)\Upsilon(2b\sigma)\Upsilon(-2ibk')}{\Upsilon(b(\sigma+ik+ik'))\Upsilon(b(\sigma-ik-ik'))\Upsilon(b(\sigma+ik-ik'))\Upsilon(b(\sigma-ik+ik'))} \end{aligned} \tag{22}$$

Let us note that a factor of the form $e^{\xi x^2}$ contributing to $\Upsilon(x)$ will always cancel in the three point formula between numerator and denominator. This is simply because

$$(\sigma + ik + ik')^2 + (\sigma - ik - ik')^2 + (\sigma + ik - ik')^2 + (\sigma - ik + ik')^2 = (2\sigma)^2 + (-2ik)^2 + (-2ik')^2,$$

the cross terms all canceling. On the other hand, a factor of the form $e^{\xi x}$ does not cancel between numerator and denominator, and it yields a net factor $e^{b\xi(-2ik-2ik'-2\sigma)}$.

We now substitute Eq. 14 in the Eq. 22. Gathering the contributions of the factors multiplying the exponential in Eq. 14 and setting the $b^2 = 0$ in the gamma functions in the square brackets leads to the zeroth approximation

$$
\begin{aligned}
C_0(k,\sigma,k') &= \frac{1}{b}\left[\pi\mu b^{-2b^2}\right]^{-\sigma-ik-ik'} b^{-2(b^2-1)(\sigma+ik+ik')} \\
&\quad\times\frac{\Gamma(\sigma+ik+ik')\Gamma(\sigma-ik-ik')\Gamma(\sigma+ik-ik')\Gamma(\sigma-ik+ik')}{\Gamma(-2ik)\Gamma(2\sigma)\Gamma(-2ik')} && (23)\\
&= \frac{1}{b}\left[\frac{b^2}{\pi\mu}\right]^{\sigma+ik+ik'}\frac{|\Gamma(\sigma+ik+ik')\Gamma(\sigma+ik-ik')|^2}{\Gamma(-2ik)\Gamma(2\sigma)\Gamma(-2ik')} && (24)
\end{aligned}
$$

This is to be compared to Eq. (12) of Ref. [5] after translating the parameters of that work to the ones used here:

$$
g=b\sqrt{2\pi},\qquad m=b\sqrt{\mu\pi},\qquad \alpha=2\sigma,\qquad k',k''\to 2k,2k'\ . \tag{25}
$$

In this dictionary quantities on the left are those of Ref. [5] and those on the right are those of the current work. Then the formula from the older work becomes

$$
\langle 2k'|e^{2g\sigma q}|2k\rangle = \frac{1}{2(2\pi)^{3/2}b}\left[\frac{b^2}{\pi\mu}\right]^{\sigma}\frac{|\Gamma(\sigma+ik+ik')\Gamma(\sigma+ik-ik')|^2}{\Gamma(2\sigma)|\Gamma(-2ik)\Gamma(-2ik')|}. \tag{26}
$$

We see that the two formulae agree up to a phase redefinition of the initial and final states and an overall normalization constant independent of the states, operator, and coupling constant.

Next we turn to the perturbative corrections to the zeroth order result. These are contained in the ratio of gamma functions inside the square brackets as well as the contribution of the exponentials in Eq. 14 to the ratio of upsilon functions in (22). Define

$$
\begin{aligned}
\Phi_{2n+1} &\equiv \phi_{2n+1}(2\sigma)+\phi_{2n+1}(-2ik)+\phi_{2n+1}(-2ik') \\
&\quad -2\mathrm{Re}\ \phi_{2n+1}(\sigma+ik+ik')-2\mathrm{Re}\ \phi_{2n+1}(\sigma+ik-ik'). && (27)
\end{aligned}
$$

Then the perturbative corrections are contained in a factor

$$
\begin{aligned}
F &= \left[\frac{\Gamma(1+b^2)}{\Gamma(1-b^2)}\right]^{-\sigma-ik-ik'}\exp\left\{b^2\gamma\Phi_1+\sum_{n=1}^{\infty}b^{4n+2}\frac{\zeta(2n+1)}{2n+1}\Phi_{2n+1}\right\} && (28)\\
&= \exp\left\{b^2\gamma(\Phi_1+2(\sigma+ik+ik'))+\sum_{n=1}^{\infty}b^{4n+2}\frac{\zeta(2n+1)}{2n+1}\left(\Phi_{2n+1}+2(\sigma+ik+ik')\right)\right\} && (29)
\end{aligned}
$$

In the weak coupling calculations of [5] the factor F was evaluated through order $O(g^{10})$, and it is of interest to compare those results with the exact proposal. So we work out the first three Φ's:

$$
\begin{aligned}
\Phi_1 &= -2ik-2ik'-2\sigma && (30)\\
\Phi_3 &= -6(k^2-k'^2)^2-12\sigma(\sigma-1)(k^2+k'^2)-6\sigma^4+4\sigma^3+i8(k^3+k'^3) && (31)\\
\Phi_5 &= 20(k^2-k'^2)^2(k^2+k'^2)-10(k^2-k'^2)^2-20\sigma^6+28\sigma^5-10\sigma^4-32ik^5-32ik'^5 \\
&\quad -20\sigma^2(1-\sigma)^2(k^2+k'^2)-20\sigma(1-\sigma)(k^4+6k^2k'^2+k'^4) && (32)
\end{aligned}
$$

We see that the first term in the exponent cancels so that F simplifies to

$$
F=\exp\left\{\sum_{n=1}^{\infty}b^{4n+2}\frac{\zeta(2n+1)}{2n+1}\left(\Phi_{2n+1}+2(\sigma+ik+ik')\right)\right\}. \tag{33}
$$

Thus we immediately confirm the structure found in [5] that through $O(b^{10})$ the only nonvanishing terms are $O(b^6)$ and $O(b^{10})$. In fact, after translating the parameters from the older work to the ones in the current work, the values of those terms agree exactly with what we have found here.

The weak coupling calculational program given in [5] can of course be carried out to any desired order. For the three point function we have the exact answer so perturbation theory gives no additional information. However the main virtue of perturbation theory is that it can be applied to an arbitrarily complicated process. Thus it can give complementary information to bootstrap approaches.

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